BARBERSHOP CLUB

# Problem

There is a barbershop club that looks like this:

1. if a member X of the club cut the hair of some member of the club, then all members of the club cut X’s hair
2. two club members are named Giulio and Cesare
3. Giulio cut cesare’s hair

Question: Did Cesare cut his own hair?

# Formalization

M = member

C = cut hair

g = guido

c = cesare

1. ∀x[ M(x) ^ ∃y (M(y)^C(x,y)) → ∀z(M(z) → C(z,x)) ]
2. M(g) ^ M(c)
3. C(g,c)
4. C(c,c) I have to negate it ¬ C(c,c), add it to the other hypothesis and check satisfiability

2)-3)-4) are ground sentences

1) is not even universal.

First we need to get in prenex normal form, moving the quantifiers out of

## 

## Prenex Normal Form for 1)

∀x[ M(x) ^ ∃y (M(y) ^ C(x,y)) → ∀z(M(z) → C(z,x)) ]

Move the quantifier outside, using **A → ∀xB = ∀x(A → B)**

**↓**

∀x ∀z[ M(x) ^ ∃y (M(y) ^ C(x,y)) → (M(z) → C(z,x)) ]

Move the quantifier outside, using **A ^ ∃xB = ∃x(A ^ B)**

**↓**

∀x ∀z[ (∃y (M(x) ^ (M(y) ^ C(x,y)) → (M(z) → C(z,x)) ]

**∃y A → B = ∀ y (A→ B)**

↓

∀x ∀z ∀y[ ( (M(x) ^ (M(y) ^ C(x,y)) → (M(z) → C(z,x)) ]

this is a universal formula, don’t need skolemization

no existential quantifiers

## Matrix in CNF

∀x ∀z ∀y[ ( (M(x) ^ (M(y) ^ C(x,y)) → (M(z) → C(z,x)) ]

A → B = ¬ A v B

∀x ∀z ∀y[ ¬ ( M(x) ^ M(y) ^ C(x,y)) v (M(z) → C(z,x)) ]

∀x ∀z ∀y[ ¬ ( M(x) ^ M(y) ^ C(x,y)) v ¬ M(z) v C(z,x) ]

remove the brackets with consecutive v

∀x ∀z ∀y[ ¬ M(x) v ¬ M(y) v ¬ C(x,y) v ¬ M(z) v C(z,x) ]

this in CNF

In conclusion we have 5 clauses to instantiate

M(g) is ground

M(c) is ground

C(g,c) is ground

¬ C(c,c) is ground

¬ M(x) v ¬ M(y) v ¬ C(x,y) v ¬ M(z) v C(z,x) no ground 2 possibilities for x, 2 for y 2 for z in total 8 possible instances

5

x = c, y = c, z = c

¬ M(c) v ¬ M(c) v ¬ C(c,c) v ¬ M(c) v C(c,c) tautological can be removed

6

x= c, y= c, z = g

¬ M(c) v ¬ M(c) v ¬ C(c,g) v ¬ M(g) v C(c,c) one ¬ M(c) can be removed

7

x = c, y=g, z = c

¬ M(c) v ¬ M(g) v ¬ M(c) v ¬ C(c,c) v C(g,c)

8

x = c, y = g, z = c

¬ M(c) v ¬ M(g) v ¬ M(g) v ¬ C(c,g) v C(g,c)

9

x≡ g, y = c, z= c

¬ M(g) v ¬ M(c) v ¬ M(c) v ¬ C(g,c) v C(g,c) one ¬ M(c) can be removed

10

x = g, y =c, z = g

¬ M(g) v ¬ M(c) v ¬ M(g) v ¬ C(g,g) v C(c,g)

11

x = g, y = g, z= c

¬ M(g) v ¬ M(g) v ¬ M(c) v ¬ C(g,c) v C(g,g) one ¬ M(g) can be removed

12

x =g, y =g, z = g

¬ M(g) v ¬ M(g) v ¬ M(g) v ¬ C(g,g) v C(g,g) tautological can be removed

1. M(g)
2. M(c)
3. C(g,c)
4. ¬ C(c,c)
5. removed
6. ¬ M(c) v ¬ M(c) v ¬ C(c,g) v ¬ M(g) v C(c,c)
7. ¬ M(c) v ¬ M(g) v ¬ M(c) v ¬ C(c,c) v **C(g,c)**
8. ¬ M(c) v ¬ M(g) v ¬ M(g) v ¬ C(c,g) v **C(g,c)**
9. ¬ M(g) v ¬ M(c) v ¬ M(c) v ¬ C(g,c) v C(g,c) also this is subsumed
10. ¬ M(g) v ¬ M(c) v ¬ M(g) v ¬ C(g,g) v C(c,g)
11. ¬ M(g) v ¬ M(g) v ¬ M(c) v ¬ C(g,c) v C(g,g)
12. removed

7 and 8 are subsumed by 3. C(g,c) must be true

# Resolution

Apply resolution with clause 1 and 6

M(g) ¬ M(c) v ¬ M(c) v ¬ C(c,g) v ¬ M(g) v C(c,c)

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6.1 ¬ M(c) v ¬ C(c,g) v v C(c,c)

also with 9. 10 11.

9.1 ¬ M(c) v ¬ C(g,c) v C(g,c)

10.1 ¬ M(c) v ¬ C(g,g) v C(c,g)

11.1 ¬ M(c) v ¬ C(g,c) v C(g,g)

resolution with 2. M(c)

6.2 ¬ C(c,g) v C(c,c)

9.2 ¬ C(g,c) v C(g,c)

10.2 ¬ C(g,g) v C(c,g)

11.2 ¬ C(g,c) v C(g,g)

resolution with 3. C (g,c)

6.3 C(c,c)

9.3 C(g,c)

10.3 ¬ C(g,g) v C(c,g)

11.3 C(g,g)

resolution with 4.¬ C(c,c) and 6.3

empty clause

UNSAT

# CDCL

More simple

can propagate literal 1. 2. 3. 4.

C(g,c) with clause 9

M(g), M(c), C(g,c), ¬ C(c,c)

C(g,c)

propagated with clause 6

¬ C(c,g)

M(g), M(c), C(g,c), ¬ C(c,c)

C(g,c) 9

¬ C(c,g) 6

¬ C(g,g) 10

Conflict with 11

Since a conflict clause without decided literal we get unsat

In conclusion the problem has answer **YES**

because negating the thesis we got UNSAT